On orthogonality of alinear quasigroups

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Definition

Two Latin squares defined on the set $\{x_1, x_2, \dots, x_m\}$ are called orthogonal if when one is superimposed upon the other every ordered pair of symbols $x_1, x_2, \dots, x_m$ occurs once in the resulting square [2].

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Definition

Binary groupoids (Q, A) and (Q, B) are called orthogonal if the system of equations

$$\begin{cases} A(x, y) = a \\ B(x, y) = b \end{cases}$$

has an unique solution (x_0, y_0) for any fixed pair of elements $a, b \in Q$ [4].

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Let $(Q, +)$ be a quasigroup. A permutation $\overline{\varphi}$ of the set Q is called an anti-automorphism of quasigroup $(Q, +)$, if the equality $\overline{\varphi}(x + y) = \overline{\varphi}y + \overline{\varphi}x$ is true for all $x, y \in Q$.

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Denote by the letter I the following anti-automorphism of a group $(Q, +)$: $I(x) = -x$ for any $x \in Q$. It is well known that $I^2 = \varepsilon$. Any anti-automorphism $\overline{\psi}$ of the group $(Q, +)$ can be represented in the form $\overline{\psi} = I\psi$, where $\psi \in Aut(Q, +)$.

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Definition

Quasigroup $(Q, \cdot)$ of the form $x \cdot y = \overline{\varphi}x + \overline{\psi}y + a$, where $(Q, +)$ is a group, a is a fixed element of the set Q, and $\overline{\varphi}, \overline{\psi} \in Aaut(Q, +)$, is called alinear quasigroup (over the group $(Q, +)$).

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A binary groupoid $(G, \circ)$ is isotopic image of a binary groupoid $(G, \cdot)$, if there exist permutations α, β, γ of the set G such that $x \circ y = \gamma^{-1}(\alpha x \cdot \beta y)$. The ordered triple of permutations (α, β, γ) of the set G is called an *isotopy* [1].

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Lemma

Suppose that finite left (right) linear (alinear) quasigroup $(Q, \cdot)$ and finite left (right) linear (alinear) quasigroup $(Q, \circ)$ have the forms $x \cdot y = \alpha x + \beta y + c$ and $x \circ y = \gamma x + \delta y + d$ over a group $(Q, +)$. Then without loss of generality for the study of orthogonality of these quasigroups we can take $c = d = 0$ [3].

Results

Orthogonality of alinear quasigroups

Theorem

An alinear quasigroup $(Q, \cdot)$ *of the form* $x \cdot y = I\alpha x + I\beta y + c$ *and an alinear quasigroup* $(Q, \circ)$ *of the form* $x \circ y = I\gamma x + I\delta y + d$*, both defined over a group* $(Q, +)$*, where* $\alpha, \beta, \gamma, \delta \in Aut(Q, +)$*, are orthogonal if and only if the mapping* $(I\delta^{-1}\gamma + \beta^{-1}\alpha)$ *is a permutation of the set* Q*.*

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Results

Parastroph orthogonality of alinear quasigroups

Theorem

For an alinear quasigroup (Q, A) of the form $A(x, y) = I\varphi x + I\psi y + c$ over a group $(Q, +)$ the following equivalences are true:

1. $A \perp A^{12} \iff$ *the mapping* $(\psi^{-1}\varphi - J_t\varphi^{-1}\psi)$ *is a permutation of the set* Q *for any* $t \in Q$;
2. $A \perp A^{13} \iff$ *the mapping* $(\varphi - J_{\psi t}J_c)$ *is a permutation of the set* Q *for any* $t \in Q$;
3. $A \perp A^{23} \iff$ *the mapping* $(\varepsilon + I\psi J_t)$ *is a permutation of the set* Q *for any* $t \in Q$;

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1. $A \perp A^{123} \Longleftrightarrow$ *the mapping* $(\psi^2 - \varphi J_{\psi^{-1}c})$ *is a permutation of the set* Q*;*
2. $A \perp A^{132} \Longleftrightarrow$ *the mapping* $(\psi - \varphi^2)$ *is a permutation of the set* Q*.*

Corollary

Any alinear quasigroup over the group S_n $(n \neq 2; 6)$ *is not orthogonal to its*
(i) (12)-*parastrophe;*
(ii) (13)-*parastrophe;*
(iii) (23)-*parastrophe.*

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